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Algebraically special hypersurface-homogeneous Einstein spaces in general relativity

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This paper illustrates the value of the Newman–Penrose complex null tetrad formalism by using it to obtain all algebraically special Einstein spaces admitting three-parameter groups of motions acting on timelike surfaces containing the repeated principal null direction. Taken together with earlier work, this enables us to give a complete list of Einstein spaces which are both algebraically special and hypersurface-homogeneous or homogeneous.

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1. Introduction

It is a pleasure to dedicate this paper to Roger Penrose. It is also, in our view, very appropriate: the main technique, the Newman–Penrose complex null tetrad method, comes from one of his best known and most widely used papers [1]; the spinor formulation of the Petrov classification of the Weyl tensor was discussed in another paper of his [2]; the spaces considered, since they are Einstein spaces, are related to LeBrun's Einstein bundle [3], which arises in Penrose's twistor theory [4] (though we have not attempted to make this link explicit); finally, if we may be forgiven for teasing our dedicatee, this paper has one feature in common with some of his best known work—namely, a gestation period measured in decades! [5].

Homogeneous and hypersurface-homogeneous spacetimes in general relativity

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are characterized by admitting a group of isometries G_r with r parameters, where $3 \leqslant r \leqslant 10$, and if $r=3$ we must add that the group acts on three-dimensional orbits. This apparently large set of possibilities rapidly reduces to the consideration of just the G_3 case: Einstein spaces with larger groups are already known (see ref. [6], section 9.2, for details and references), and are listed in section 9.

Thus to complete a study of homogeneous or hypersurface-homogeneous algebraically special Einstein spaces we have only to consider those admitting a group of motions G_3, which we can take to be simply transitive (relevant multiply transitive possibilities have been considered in refs. [7–9]). There are two possibilities: either the homogeneous hypersurfaces contain the repeated principal null direction (PND), or they do not. The second case was considered earlier by one of us [10] (with some help from the other), and the new work described in the present paper complements ref. [10] by considering the first case, which is a subset of Kundt's class [6, ch. 27].

Related solutions with less symmetry have been considered by Kramer [11], who considered cases with a null Killing vector; Bampi and Cianci [12], who took a G_2 acting on null surfaces *; and Hoffman [13], who considered a G_2 on timelike surfaces (i.e. stationary axisymmetric spacetimes). All these authors solved only the vacuum case. Barnes [9] considered a G_3 acting on null two-surfaces and gave the general Einstein space solution. Harness [14] found some algebraically special solutions in the course of his investigation of metrics with a G_3 acting on timelike hypersurfaces. Hoffman's S-class and the first set of solutions of Bampi and Cianci are special cases of Kramer's. Finally, some of the Petrov type N solutions in the present class are "Lobatchevski plane gravitational waves" [15], which can be defined as non-vacuum Petrov type N spaces which are conformal to pp-waves.

Because all G_3 except those of types VIII and IX contain an Abelian G_2, almost all of our solutions have two commuting Killing vectors, and so can be considered as examples of stationary axisymmetric or cylindrically symmetric spacetimes, or Bampi and Cianci's null surface analogues. However, unlike the spatially homogeneous cases (see ref. [10]) or the cases with genuine axes of symmetry (see ref. [6], section 17.2), the Abelian G_2 need not act on orthogonally transitive two-surfaces.

When the homogeneous hypersurfaces contain the repeated PND, they could be null or timelike. The null hypersurface case is considered in ref. [6], section 21.2, and leads only to plane waves and spaces of constant curvature. Thus our task here is to obtain all algebraically special Einstein spaces with homogeneous timelike hypersurfaces containing the repeated PND. In section 2 we set out the system of equations to be used; in section 3 we integrate as far as possible without considering the possible Petrov types separately; in sections 4–7 we consider the

* We intend to publish elsewhere, in collaboration with G.C. Joly, a note on these metrics.

different Petrov types in turn, and in section 8 we give the form of the metric. Some of the more tedious details are omitted. Section 9 contains a full list of the solutions possible, including those obtained elsewhere, and some concluding remarks.

2. Setting up the problem

2.1. CHOICE OF THE TETRAD

Let $(\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}})$ be a Newman–Penrose null tetrad, with corresponding differential operators $(D, \Delta, \delta, \bar{\delta})$ (see ref. [6]), where $\boldsymbol{k}$ is the repeated principal null direction so that $\Psi_0 = \Psi_1 = 0$. At one point in each homogeneous hypersurface, we choose $\boldsymbol{l}$ and $\boldsymbol{m} - \bar{\boldsymbol{m}}$ to be tangent to the hypersurface, so that $\boldsymbol{m} + \bar{\boldsymbol{m}}$ is normal to it. These vectors are then dragged round the hypersurface by the action of the group. This implies that all the scalars derived from the tetrad vectors (the spin coefficients and the Riemann tensor components, for example) are functions only of a coordinate x which labels different homogeneous hypersurfaces.

Using this condition in the commutators [1] provides the following relations:

$$\begin{aligned}\mathrm{Im}(\rho+2\epsilon) &= \mathrm{Im}(\mu-\lambda-2\gamma) = \mathrm{Re}(\tau+\bar{\pi}) \\ &= \mathrm{Re}(\rho) = \mathrm{Re}(\mu+\lambda) = \mathrm{Im}(\alpha+\beta) = 0 \,. \end{aligned} \tag{2.1}$$

The first three of these relations come from the conditions that D, Δ, and $\delta - \bar{\delta}$ are surface forming and the other three from the hypersurface orthogonality of $\delta + \bar{\delta}$. * The commutators of D, Δ, and $\delta - \bar{\delta}$ can be used to classify the group into one of the Bianchi types in the usual way (see ref. [6], section 8.2).

We still have the freedom to use an x-dependent null rotation about $\boldsymbol{k}$ in the plane of $(\boldsymbol{k}, \boldsymbol{m} - \bar{\boldsymbol{m}})$ and an x-dependent boost in the $(\boldsymbol{k}, \boldsymbol{l})$ plane. Under the null rotation, $\lambda + \mu$ transforms by

$$\lambda + \mu \to \lambda + \mu + 2\mathrm{i}u(\alpha+\beta) + 2\mathrm{i}\delta u \,,$$

where u is the (real) "angle" of the null rotation. Under the boost,

$$\alpha + \beta \to \alpha + \beta + \delta(\log A) \,,$$

where A is the (real) boost parameter. Since already, according to (2.1), $\lambda + \mu$ is imaginary and $\alpha + \beta$ is real, these freedoms can be used to set

$$\lambda + \mu = 0 \,, \qquad \alpha + \beta = 0 \,. \tag{2.2}$$

Note that we cannot use the tetrad freedom to put the reciprocal group com-

* From the point of view of the commutators, it is not known that $\delta + \bar{\delta}$ is orthogonal to Δ, D and $\delta - \bar{\delta}$, so the first three conditions are independent of the second three.

mutators into canonical form because we have insisted on using the principal null direction as one of the basis vectors ★.

2.2. STRUCTURE OF THE NEWMAN–PENROSE EQUATIONS

To understand how to proceed, it helps to examine the structure of the Newman–Penrose equations. If one writes down the definitions of connection and curvature in Cartan's form,

$$d\omega^i = -\omega^i{}_j \wedge \omega^j \,, \tag{2.3}$$

$$d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j = R^i{}_j \,, \tag{2.4}$$

then the integrability conditions of the first set, eqs. (2.3), i.e. $d^2\omega^i = 0$, yield the first Bianchi identities,

$$(d\omega^i{}_j + \omega^i{}_k \wedge \omega^k{}_j) \wedge \omega^j = 0 \,, \tag{2.5}$$

which can equally well be thought of as the Jacobi identities for the basis of tangent vectors dual to $\{\omega^i\}$. The integrability conditions of (2.4), $d^2\omega^i{}_j = 0$, yield the (second) Bianchi identities,

$$dR^i{}_j - R^i{}_k \wedge \omega^k{}_j + \omega^i{}_k \wedge R^k{}_j = 0 \,. \tag{2.6}$$

The conditions (2.5) give 16 real equations, while eqs. (2.4) define the 20 independent components of the Riemann tensor in terms of the connection. In the Newman–Penrose formalism, these 36 real equations become the 18 complex equations labelled (4.2a)–(4.2r) in ref. [1], which we will refer to generically as the NP equations (although they are not the only equations in ref. [1]!) or separately by the names (NPa)–(NPr), while eqs. (2.3) are the commutators, eqs. (4.4) in ref. [1], and the conditions (2.6) are the Bianchi identities, eqs. (4.5) in ref. [1] (for Einstein spaces) or eqs. (7.61)–(7.71) in ref. [6] (for general energy–momentum).

If one starts with a co-frame $\{\omega^i\}$, then the conditions (2.5) are automatically satisfied because of the definition (2.3). However, if one starts with a set of spin coefficients, then (2.5) will give restrictions which ensure integrability, i.e. the existence of a co-frame. In particular, if we use the basis freedom to choose the value of some of the spin coefficients, then further consequences will follow from eq. (2.5), or equivalently, from the combinations of the NP equations which do not involve the Riemann tensor components. Thus, if we use the freedom of basis choice to put the spin coefficients in a nice form, it is (NPa)–(NPr) which will probably be best used first.

Similarly, if one starts with a set of spin coefficients, then (2.6) are automati-

★ In any case, the canonical forms are not quite so convenient in the timelike case, when the isotropy group is SO(2,1) instead of SO(3) [16].

cally satisfied, because of definition (2.4). However, if we place restrictions directly on the Riemann tensor components (by choosing the Petrov type, or the tetrad components of the curvature), then further consequences follow from the Bianchi identities (2.6). In particular, if eq. (2.4) is no longer identically satisfied, then the integrability condition $d^2R^i{}_j=0$ for (2.6) may provide non-trivial relations (which Brans [17] refers to as "post-Bianchi identities"; Kinnersley [19] used them in the form of commutators applied to Ψ_2). Thus if one begins by putting the Riemann tensor in a nice form, it is the Bianchi identities which are most likely to yield helpful information ⋆.

Using the tetrad freedom to eliminate Riemann tensor components is usually a good plan, since it often fixes the frame completely, whereas choosing spin coefficients often leaves a freedom of some constant Lorentz transformations.

3. Integrating the equations

In the present problem, the restriction $\Psi_0=\Psi_1=0$ substituted into the Bianchi identities immediately yields $\kappa=\sigma=0$ (which is of course the Goldberg–Sachs theorem) and this together with the earlier restrictions (2.1) on the spin coefficients, gives $\rho=0$, $\epsilon=\bar{\epsilon}$ on using (NPa). The specializations so far can be summarized as

$$\begin{aligned} &\Psi_0=\Psi_1=0\,, &&\alpha=-\beta\,, &&\mathrm{Im}(\epsilon)=0\,,\\ &\kappa=0\,, &&\lambda=-\mu\,, &&\mathrm{Im}(\mu)=\mathrm{Im}(\gamma)\,,\\ &\rho=0\,, &&\sigma=0\,, &&\mathrm{Re}(\tau+\pi)=0\,, \end{aligned} \tag{3.1}$$

and we have to solve for β, γ, μ, ν, π and τ, and, of course, the remaining Ψ_4.

In the appendix we give the form of the remaining non-trivial NP equations, Bianchi identities, and commutators with these specializations: the commutators are written in terms of D, Δ, $\delta-\bar{\delta}$, and $\delta+\bar{\delta}$ since the first three of these are the reciprocal group generators, and the non-trivial Bianchi identities are numbered (B3)–(B8), according to their position in eq. (4.5) of ref. [1].

First we show that $\epsilon=0$ in all cases except Petrov type D. (NPh) + (NPp) + (NPq) − (NPg) gives

$$\pi\bar{\pi}-\pi^2-\tau\bar{\tau}+\tau^2+2(\pi+\tau)(\beta+\bar{\beta})=4\epsilon\mu\,, \tag{3.2}$$

while (B4) + (B5) is

$$(\tau+\pi)\Psi_2=2\epsilon\Psi_3\,. \tag{3.3}$$

We need only consider the case $\Psi_2\neq 0$, since in Petrov types III and N the Bianchi

⋆ Related remarks about the division of the Newman–Penrose equations into definitions and integrability conditions appear in a series of papers by Papapetrou, e.g. ref. [18].

identities (B5) or (B7), respectively, imply (trivially) that $\epsilon=0$. Equation (3.3) means, according to (3.1), that Ψ_2 is an imaginary multiple of Ψ_3. Furthermore, it follows from (NPp) − (NPq) − (NPg) − (NPh), on using the fact that $\tau+\pi$ is imaginary to eliminate $\bar{\tau}$, that

$$2\delta\log(\tau+\pi) = -(\pi+\bar{\pi}) \; .$$

A comparison with the real part of (NPe) shows that, if $\epsilon\neq 0$, then $(\tau+\pi)/\epsilon$ is constant. We can therefore use the remaining null rotation freedom

$$\Psi_3 \to \Psi_3 + 3\mathrm{i}u\Psi_2 \tag{3.4}$$

to set $\Psi_3=0$. Then (B6) implies $\mu=0$, in which case $\epsilon=0$ by (B7) and the argument is complete.

The type D metrics with $\epsilon\neq 0$ are all well known, being generalizations of Kinnersley's vacuum cases [19] to $\Lambda\neq 0$, or solutions of the Robinson–Bertotti form given as eq. (10.8) in ref. [6], and will not be considered in detail here.

We will therefore assume henceforth that $\epsilon=0$.

The real part of (3.2) together with (3.1) yields the following important relation:

$$(\tau+\pi)(\tau+\bar{\pi}) = 0 \; , \tag{3.5}$$

which is one of the Jacobi identities for D, Δ and $\delta-\bar{\delta}$.

There are two further relations which can be conveniently derived at this stage. The first, which contains no derivatives or curvature tensor components, follows from (NPi) + (NPo) + (NPr), on using eqs. (3.1):

$$0=\gamma(\pi+\tau+2\beta-\bar{\beta}-\bar{\tau})-\bar{\gamma}(\pi+\beta)+\mu(\beta-2\bar{\pi})+3\bar{\mu}\beta \; . \tag{3.6}$$

The real part is

$$\mu(2\bar{\beta}-\bar{\pi}+\tau+\pi)+\bar{\mu}(2\beta+\bar{\pi}+\bar{\tau}-\pi)=0 \; , \tag{3.7}$$

and the imaginary part, which is another Jacobi identity, is

$$(\mu+\bar{\mu})[(\pi-\bar{\pi})+2(\beta-\bar{\beta})]+(\gamma+\bar{\gamma})[(\tau-\bar{\tau})-2(\beta-\bar{\beta})]=0 \; . \tag{3.8}$$

The second equation comes from (NPm) + (NPo) − (NPr), making use of (3.7):

$$2\delta(\gamma-\mu) = -(\pi+\bar{\pi})(\gamma-\mu)+(\bar{\mu}-\mu)(\tau+\pi) \; , \tag{3.9}$$

which can be integrated if μ is known. Note that the imaginary part of this is consistent with $\mu-\gamma$ being real.

The next problem is to choose a suitable coordinate x to label the hypersurfaces. A common choice in the NP method is to take $D=\partial_u$ and set $\rho=-(u+\mathrm{i}a)^{-1}$, which in most cases can be used to define u uniquely. This will not work in the present setting since u would be a coordinate in the surfaces we want to label and anyway $\rho=0$. We proceed as follows.

First let the function f be defined by

$$\delta \log f = \beta + \bar{\beta} , \tag{3.10}$$

so that f is unique up to a multiplicative constant. Then let

$$\delta + \bar{\delta} = 2f \partial_x . \tag{3.11}$$

Given f, this defines x up to an additive constant, and scaling f produces a corresponding scaling of x. We can write (NPg) in the form

$$(\pi/f)' = -(\pi/f)^2 ,$$

where the prime means differentiation with respect to x. Provided $\pi \neq 0$, this integrates to

$$\pi = f/(x + \mathrm{i}a) , \tag{3.12}$$

where a is a constant which is chosen to be real by using up the freedom to add a constant to x. When eq. (3.12) is satisfied, (NPp) is identically satisfied in both of the cases implied by eq. (3.5).

We have assumed above that $\pi \neq 0$. But if $\pi = 0$, eq. (3.5) shows that $\tau = 0$ and (NPf, h, i) show that the solution must be vacuum type N. We shall deal with this case separately in section 7.5. From the Bianchi identities, it also follows easily that this case is the only one (except for type D) in which the leading non-zero Ψ_A can be constant.

We can now integrate (B4) for Ψ_2 (recall that we are assuming $\epsilon = 0$):

$$\Psi_2 = (m + \mathrm{i}l)/(x + \mathrm{i}a)^3 , \tag{3.13}$$

for some real constants m and l. We have assumed that $\tau = -\pi$ since otherwise eq. (3.3) implies that $\Psi_2 = 0$. (In other words, the case $\tau + \pi \neq 0$ is included by setting $m = l = 0$.)

It is straightforward to integrate (NPl) for β, using (3.13), (3.10) and (3.11):

$$f\beta = -\frac{\Lambda(x + \mathrm{i}a)}{2} - \frac{m + \mathrm{i}l}{4(x + \mathrm{i}a)^2} + \mathrm{i}\beta_1 , \tag{3.14}$$

where β_1 is a constant. Comparing this with $2ff' = 2f(\beta + \bar{\beta})$ gives a general form for f^2:

$$f^2 = f_0 - \Lambda(x^2 + a^2) + \frac{m + \mathrm{i}l}{2(x + \mathrm{i}a)} + \frac{m - \mathrm{i}l}{2(x - \mathrm{i}a)} , \tag{3.15}$$

where f_0 is a (real) constant.

(NPh) can be written

$$-(f\pi)' = \pi\bar{\pi} + \Psi_2 + 2\Lambda ,$$

which is only consistent with (3.15) if β_1 is real and

$$2af_0 = -l\,. \tag{3.16}$$

The only quantities which are still (in principle) undetermined by the equations of this section are μ, ν, Ψ_3 and Ψ_4. We can find μ by integrating (NPm) + (NPi), and Ψ_3 then follows from (NPi). In general, ν can be found by integrating (NPn), in which case Ψ_4 follows from (NPj), and (B8) is then automatically satisfied. However, if $\Psi_2 = 0$, (B8) can be directly integrated for Ψ_4, and ν can then usually be obtained without further integration by eliminating $\delta\nu$ from (NPj) and (NPn). It is convenient to define two real functions $N(x)$ and $n(x)$ by

$$f\nu = N + \mathrm{i}n\,. \tag{3.17}$$

Once the constraints (3.2), (3.6) and (3.16) are satisfied, the only remaining constraint comes from (NPf).

The only remaining freedom in the tetrad and x coordinate is a constant null rotation [with imaginary parameter, as in (3.4)], a constant boost, and the scaling of x (and correspondingly of f) by a constant multiple.

The Bianchi type can be determined from quantities a_i and n^{ij} defined in ref. [6], which can be easily calculated from the commutators given in the appendix to this paper. The only non-zero components in the triad we are using here are

$$\begin{aligned} &a_2 = \mu - \gamma\,, && a_3 = -\mathrm{i}(\tau + \pi)/\sqrt{2}\,, \\ &n^{11} = \mathrm{i}(\bar{\nu} - \nu)/\sqrt{2}\,, && n^{13} = -(\mu + \bar{\gamma})\,, \\ &n^{12} = \mathrm{i}(2\beta - 2\bar{\beta} - \tau - \bar{\pi})/\sqrt{2}\,, && n^{33} = \mathrm{i}\sqrt{2}\,(\tau + \bar{\pi})\,. \end{aligned} \tag{3.18}$$

To integrate further we have to consider the two cases implied by (3.5) separately. We then consider each Petrov type in turn and finally give a metric form which covers all the possibilities found except the type D forms with $\epsilon \neq 0$ (see section 5) and the type N vacua (see section 7.5). The various new constants which have been introduced have to be evaluated in the different cases.

To see that the constants in the solution are essential in each case we only have to consider the residual constant boost or null-rotation freedom, and rescaling of the x coordinate, as our frame is otherwise invariantly defined, unless the solution admits several distinct G_3's (which would be subgroups of a G_r, $r > 3$). This happens if either the solution is homogeneous or the hypersurfaces of homogeneity admit an isotropy; the latter can only arise as a boost in Petrov type D or a null rotation in Petrov type N. As a check on this, and to see where the (known) type III and N homogeneous Einstein spaces arise within our class of solutions, we have used the classification methods described (e.g.) in ref. [20] as implemented in the computer algebra system CLASSI [21].

3.1. THE CASE $\tau+\pi=0$

We can include the cases where π is real in this section by setting $a=0$.

The constraint (3.2) is identically satisfied, and (NPf) is satisfied provided

$$f_0=4a(\beta_1-a\Lambda)\ . \tag{3.19}$$

This, together with (3.16), implies that $l=8a^2(a\Lambda-\beta_1)$. Equation (B7) − (B6) can be integrated immediately, using eq. (3.12):

$$\Psi_3=(s+\mathrm{i}t)/(x+\mathrm{i}a)^3\ , \tag{3.20}$$

where s and t are real constants of integration. (B7) + (B6) then gives

$$3\mu\Psi_2=(2\beta-\pi)\Psi_3\ . \tag{3.21}$$

We can obtain an expression for μ by integrating (NPi) − (NPm):

$$\mu f=\mu_0\frac{x+\mathrm{i}a}{x-\mathrm{i}a}-(s+\mathrm{i}t)\left(\frac{1}{6(x^2+a^2)}+\frac{1}{3(x+\mathrm{i}a)^2}\right), \tag{3.22}$$

where μ_0 is a constant of integration. This is consistent with (NPi) and (3.20) provided

$$12a^2\mu_0+s=0 \tag{3.23}$$

(so μ_0 is real) and it is consistent with (3.21) provided

$$2\mathrm{i}(s+\mathrm{i}t)(\beta_1-a\Lambda)=3(m+\mathrm{i}l)\mu_0\ . \tag{3.24}$$

One can now deduce that $tl+sm=0$. Setting $\tau+\pi=0$ in (3.9) gives an equation which can be integrated independently of μ:

$$\gamma=\mu+c(x^2+a^2)^{-1/2}\ , \tag{3.25}$$

where c is a real constant.

The only remaining constraint is the Jacobi identity (3.8), which in conjunction with (3.1) implies that either $\mu=\gamma$ or $2\beta+\pi$ is real, i.e.,

$$c\beta_1=0\ . \tag{3.26}$$

From (3.18), we see that the groups of motion are in class B (see ref. [6], section 8.2) unless $c=0$. The conditions that $\boldsymbol{k}$ is a null Killing vector are either $a=0$ or the solution is of type N.

We still have to determine ν and Ψ_4, but this can most conveniently be done for each Petrov type separately.

3.2. THE CASE $\tau+\bar{\pi}=0$

We assume in this section that π is not real (since $\pi=\bar{\pi}$ is a special case of the previous section) so that $\tau+\pi\neq 0$ and hence [according to eq. (3.3)], $\Psi_2=0=m=l$. This leads to considerable simplification in the forms for f [eq. (3.15) with $f_0=0$ by (3.16), since $a\neq 0$] and β [eq. (3.14)]. Clearly, $\Lambda<0$. (NPf) is identically satisfied, and (3.2) gives

$$\pi+\bar{\pi}-2(\beta+\bar{\beta})=0$$

(since $\pi\neq\bar{\pi}$), which is also identically satisfied.

The Bianchi identity (B6) – (B7) can be integrated, giving

$$\Psi_3=\frac{s+\mathrm{i}t}{(x+\mathrm{i}a)^2(x-\mathrm{i}a)}, \tag{3.27}$$

while (B6) + (B7) gives

$$\Psi_3(2\pi-\bar{\pi}-2\beta)=0\,. \tag{3.28}$$

As in the previous section, we can integrate (NPi) – (NPm) to get an expression for μ:

$$f\mu=\mu_0-\mathrm{i}\,\frac{s+\mathrm{i}t}{2a^2}\left(\frac{a}{x+\mathrm{i}a}+\tan^{-1}(x/a)\right), \tag{3.29}$$

which is only consistent with (NPi) provided μ_0 is real and $s=0$. Integrating (3.9) gives

$$f\gamma=\gamma_0+\frac{t}{4a^2}\left(4\tan^{-1}(x/a)+\frac{3a}{x+\mathrm{i}a}+\frac{a}{x-\mathrm{i}a}\right), \tag{3.30}$$

where γ_0 is real.

The only remaining constraints come from eq. (3.6) [cf. (3.28)]:

$$a\Lambda\gamma_0+\beta_1(\mu_0-\gamma_0)=0\,,\qquad t(\beta_1-2a\Lambda)=0\,. \tag{3.31}$$

The groups of motion are all in class B, since $\tau+\pi\neq 0$. $\boldsymbol{k}$ is a null Killing vector. As in the $\tau+\pi=0$ case, the only remaining unknowns are Ψ_4 and ν, which can be most conveniently found for each Petrov type separately.

4. Solutions of Petrov type II

We have $\Psi_2\neq 0$ which means that $\tau+\pi=0$ [eq. (3.3)]. From (3.7) we see that $\bar{\mu}(2\beta-\pi)$ is imaginary and comparison with (3.20), (3.13) and (3.21) shows that Ψ_2 is a constant imaginary (or zero) multiple of Ψ_3. The significance is that the remaining constant null rotation (3.4) can be used to set $\Psi_3=0$, in which case $\mu=0$ by eq. (3.21). Then (3.9) gives

$$\gamma=\gamma_0(x^2+a^2)^{-1/2}$$

for some real constant γ_0 with $\gamma_0\beta_1=0$ from (3.8). Equation (NPn) reduces to

$$[(x^2+a^2)N]'=-2i(\beta-\bar{\beta})(x^2+a^2)n/f\equiv 2(a+2\beta_1(x^2+a^2)f^{-2})n\,,\qquad n'=-4\beta_1 f^{-2}N\,. \tag{4.1}$$

This in general gives second order linear homogeneous equations for N and n, the latter, if $\beta_1\neq 0$, being

$$f^2[(x^2+a^2)f^2n']'+8[2\beta_1^2(x^2+a^2)+a\beta_1 f^2]n=0\,. \tag{4.2}$$

Given ν, Ψ_4 is determined by (NPj):

$$\Psi_4=(\bar{\pi}-\pi+2\bar{\beta}-6\beta)N+i(\bar{\pi}+\pi+2\bar{\beta}-6\beta)n\,. \tag{4.3}$$

If $\beta_1=0$, eq. (4.1) can be integrated:

$$\nu=\frac{\nu_0+\nu_1[2ax+i(x^2+a^2)]}{f(x^2+a^2)}\,. \tag{4.4}$$

The vacuum solutions for the special case $l=0=\beta_1$ were given by Harness [14]. The case with $\Lambda=\gamma_0=l=\nu_1=0$ is contained in the solutions given by Bampi and Cianci [12] as eqs. (2.5)–(2.9), their ϵ_0 being constant and equal to ν_0/m. The vacuum solution with $a=0$ and $\gamma_0=0$ is given as eq. (4.29) in ref. [13] and as a special case of eqs. (2.1)–(2.4) in ref. [12] *. It is among the special cases of Kramer's metric [6, section 21.4] which were considered by van Stockum [26].

If $a=0=\Lambda$, eq. (4.2) for n becomes

$$m^2x(xn')'+16\beta_1^2x^4n=0\,,$$

which is Bessel's equation of order 0 in $2\beta_1x^2/m$, and this solution is given as eq. (4.30) in ref. [13] (with m removed by scaling, and in different coordinates).

If ν is zero, we obtain Petrov type D solutions related to the type II solutions in the manner described in ref. [6], theorem 27.1.

The groups are of class A if and only if $\gamma_0=0$. The class A types are as follows: VIII if $a\beta_1\neq 0$ (and $\gamma_0=0$); VII_0 if $\beta_1=0$, $a\nu_1<0$; VI_0 if $\beta_1=0$, $a=\nu_1>0$ or $a=0$, $\beta_1\neq 0$; II if $\beta_1=0$, $a=0$ or $\beta_1=0$, $\nu_1=0$; I if $a=\beta_1=\nu_1=0$.

The class B groups ($\gamma_0\neq 0$) have group parameter h given by $h^{-1}=-(4a\nu_1/\gamma_0^2+1)$. The group is of type VII_h, VI_h, IV or III if this expression is positive, negative, zero or -1 (i.e., $\nu_1=0$), respectively.

In general, the solution has four essential parameters: Λ, m, a, the two constants in n, and either β_1 or γ_0 [see eq. (3.26)], less the freedom of a constant boost and a rescaling of x. f_0 is determined by (3.19) and l is determined by (3.16). The cases with $a=0$ have three parameters.

* In ref. [12], eq. (2.4) contains a misprint: $(\mu^2/2)$ should read $(\mu^2/2\beta^2)$.

5. Solutions of Petrov type D

Most of these solutions have already appeared above as special cases of type II. Moreover, all of them are well known. For vacuum, the ones with non-constant Ψ_2 were given by Kinnersley [19] as his type IV.A and IV.B metrics, and the Einstein space generalizations are included in the metrics of Cahen and Defrise [7], which appear as eqs. (11.19) and (11.42), and as eq. (27.41) in ref. [6], and they are all included in the forms given in ref. [22]. The case with constant Ψ_2 (which can only happen if $\pi=0$) is of the Bertotti–Robinson form; see eq. (10.8) and section 10.5 of ref. [6]. The solutions admit a G_4 with a one-parameter family of G_3 transitive subgroups if Ψ_2 is not constant, and a G_6 if Ψ_2 is constant.

We will omit the details of the recovery of the solutions, noting just the following points. If $\epsilon\neq 0\neq a$ we can set $\epsilon=\gamma$ and then $8a\epsilon^2=(l-8a^3\Lambda)/(x^2+a^2)$, unless $l-8a^3\Lambda=0$. These are the Λ generalization of Kinnersley case IV.A and the groups G_3 are of type III. The same is true if $l-8a^3\Lambda=0$, but we then have $\gamma=0\neq\epsilon$. This family of solutions also occurs as the type D specializations in section 4 with $a\neq 0$, in the form of solutions with $\epsilon=0$ and groups of type VIII, VI_h or II. The metric forms are type (B2) and (B3) in ref. [8]. If $\epsilon\neq 0=a$ the solutions are the generalizations of Kinnersley's IV.B, again with groups of type III, which are alternatively obtained with $\epsilon=0$ in terms of G_3 subgroups of types III or I (section 4 with $\beta_1=a=0\neq\gamma_0$) or VI_0 (section 4 with $a=0\neq\beta_1$). The metric forms are case (B1) in ref. [8].

6. Solutions of Petrov type III

We now have $\Psi_2=0\neq\Psi_3$. The possibilities $\tau+\pi=0$ and $\tau+\bar{\pi}=0$ can both arise. Equation (3.15) becomes

$$f^2=p^2(x^2+a^2)\,, \tag{6.1}$$

where $p^2=-\Lambda$, because f_0 vanishes for both possibilities [consider either (3.16), if $a\neq 0$, or (3.19)].

Despite there being no case with Ψ_3 constant in our standard tetrad, there is a special subcase which is four-dimensionally homogeneous, and we discuss the relation of this to the other cases in a final subsection below.

6.1. PETROV TYPE III SOLUTIONS WITH $\tau+\pi=0$

In this case $2\beta=\pi$ from (3.21), so $\beta_1=a\Lambda$. The Jacobi identity (3.8) can be written, using (3.9), as $ac=0$. (B8) can be solved for Ψ_4:

$$p^2\Psi_4 = \frac{K}{(x+ia)^3} + (s+it)\left(\frac{(s+it)}{(x+ia)^5(x-ia)} - \frac{6\mu_0}{(x+ia)^3(x-ia)} + \frac{2c\tan^{-1}(x/a)}{a(x+ia)^3}\right),$$

where K is a complex constant. A (messy) expression for ν can be obtained algebraically from Ψ_4 by solving (NPn) + (NPj); we omit the result, though we have obtained and checked it with the help of REDUCE [23].

There are three essential parameters when $a\neq 0$: Λ, a or γ_0, s and t less the coordinate rescaling. [K can be made real or imaginary by using the remaining null-rotation tetrad freedom, then scaled to 0, 1 or i by means of the remaining boost, and μ_0 is fixed by (3.23).] The group is of Bianchi type VIII.

If $a=0$ we have $s=0$ from (3.23) and we can obtain a relatively simple form for ν and Ψ_4:

$$\begin{aligned} \nu &= (\gamma_0+\mu_0)(2\mu_0/x^2+it/3x^4)/p^3+\nu_0/x^3+i\nu_1/x\,, \\ \Psi_4 &= -2p\nu_0/x^3-t^2/p^2x^6-2i(2\mu_0+\gamma_0)t/p^2x^4\,, \end{aligned} \tag{6.2}$$

where ν_0 and ν_1 are real constants and $K=-2\nu_0 p^3$. The Petrov type III four-dimensionally homogeneous case occurs when $\gamma_0+4\mu_0=0=\nu_1$.

The group is in class B if and only if $\mu_0\neq\gamma_0$. These $a=0$ solutions have the following group types: if $\mu_0+\gamma_0\neq 0$ (the general case), type VI$_h$, where

$$h=-[(\gamma_0-\mu_0)/(\gamma_0+\mu_0)]^2\,;$$

if $\gamma_0+\mu_0=0\neq\gamma_0-\mu_0$, type IV (if $\nu_1\neq 0$) or type V (if $\nu_1=0$); if $\gamma_0=\mu_0\neq 0$, type VI$_0$; and if $\gamma_0=\mu_0=0$, type II (if $\nu_1\neq 0$) or type I (if $\nu_1=0$). There are three essential parameters: Λ, t, γ_0, μ_0, and ν_1 less the boost and rescaling freedoms. The Bianchi type VI$_h$, IV, V and VI$_0$ solutions with $a=0$ were independently rediscovered by Harness [14].

6.2. PETROV TYPE III SOLUTIONS WITH $\tau+\bar{\pi}=0\neq \mathrm{Im}(\pi)$

In this case (3.31) implies $\beta_1=2a\Lambda$ and $2\mu_0=\gamma_0$. The null rotation can be used to set $\gamma_0=0$. Using (3.29) and (3.30) in the remaining equations (B8), (NPj) and (NPn), we can arrive at lengthy expressions (obtained and checked with the help of REDUCE) for ν and Ψ_4, the latter being:

$$\Lambda(x+ia)^4\Psi_4 = K(x-ia) + \frac{t^2}{a^3(x-ia)} \times\left(\frac{-3ix^3+3x^2a+21ixa^2+23a^3}{12(x-ia)} + (x+ia)^2\tan^{-1}(x/a)\right).$$

The group is of Bianchi type VI$_{-1/9}$. There are three essential parameters: Λ, a, t

and the two in Ψ_4, less the boost and rescaling freedoms. For $K=0$, we recover the four-dimensionally homogeneous case.

6.3. THE HOMOGENEOUS PETROV TYPE III SOLUTION

The four-dimensionally homogeneous Einstein space of Petrov type III appears as eq. (10.34) in ref. [6] and eq. (4.9) in ref. [10] *, it seems to have been first given by Kaigorodov [24]. Here we can obtain it as described in the previous two sections.

The full G_4 group of motions of this solution is of type $A_{4,5}^{1/4,-1/2}$ in the classification of ref. [25], and from that paper we can extract a list of the G_3 subgroups, up to conjugacy. We find groups of type $\mathrm{VI}_{-1/9}$ on surfaces which contain the repeated principal null direction (as found in section 6.2) and surfaces which do not, and groups of types $\mathrm{VI}_{-(7/3)^2}$ and I acting on surfaces containing the repeated principal null direction. These last two arise as described at the end of section 6.1, the group depending on whether $\gamma_0 \neq 0$ or $\gamma_0 = 0$. Thus we find that each type of G_3 subgroup has been recovered either here or in ref. [10], and the representations of the homogeneous case arising in the present paper each occur in a multi-parameter family of solutions with less symmetry.

7. Solutions of Petrov type N

Here Ψ_4 is the only non-zero Ψ_A. If Ψ_4 varies, both possibilities, $\tau+\pi=0$ and $\tau+\bar{\pi}=0$, arise and (6.1) then holds, as in Petrov type III. The $a=0$ solutions can be derived as limits of the general $\tau+\pi=0$ case, but this is rather messy and it is simpler to discuss them separately, which we do in section 7.3. The special case $\tau=\pi=0$, giving vacuum solutions, can also occur and is treated in section 7.5.

By the arguments in section 3, Ψ_4 is constant only if $\pi=\beta=0$, which leads to plane waves (see section 7.5). However, there is a non-vacuum four-dimensionally homogeneous solution, which in fact admits a G_5, and we discuss the relation of this to the families with a G_3 in section 7.4.

7.1. PETROV TYPE N METRICS WITH $\tau+\pi=0\neq \mathrm{Im}(\pi)$

In this case section 3.1 gives the following constraints:

$$a(\beta_1 - a\Lambda)=0\,, \qquad a^2\mu_0=0\,, \qquad c\beta_1=0\,. \tag{7.1}$$

Thus with $a\neq 0$, we have $\gamma=\mu=0$. We find

$$\Psi_4 = K/(x+\mathrm{i}a)^3\,,$$

* In both cases, with a misprint: in ref. [6] the dz dy coefficient should be $-8ue^{2z}$, and in ref. [10] there are incorrect factors of 2.

where K is a complex constant, and

$$\nu = -\frac{x^2+a^2}{24a^2 f}\left(\frac{K(x+5\mathrm{i}a)}{(x+\mathrm{i}a)^3} - \frac{\bar{K}}{(x-\mathrm{i}a)^2}\right).$$

The group is of type VIII, as in the corresponding Petrov type II and III solutions, and there is one essential parameter, which can be taken to be Λ, since a and K can be set against the null-rotation, boost and coordinate scaling freedoms. Thus for given Λ there is only one solution in this class. The metric is of type (C3) in ref. [8] and admits a G_4. It also appears as the Einstein space on the line labelled $A_{-2}(x, y)$ in table 1 of ref. [15]. By applying a null rotation (with parameter $2\mathrm{i}ap\nu$ in the coordinates of section 8) it can be transformed to the form of a solution with $\tau+\bar{\pi}=0$ and a group of Bianchi type III, the opposite sign of a and $\beta_1=0$.

7.2. PETROV TYPE N SOLUTIONS WITH $\tau+\bar{\pi}=0\neq \mathrm{Im}(\pi)$

The constraints of section 3.2 do not restrict β_1. If $\beta_1\neq a\Lambda$, we can use the null-rotation freedom to set $\mu_0=0$ and then (3.31) implies $\gamma=0$. We consider $\beta_1=a\Lambda$ separately.

From (B8) we obtain

$$\Psi_4 = \frac{K}{(x-\mathrm{i}a)^3}\left(\frac{x+\mathrm{i}a}{x-\mathrm{i}a}\right)^{-2\beta_1/a\Lambda}, \qquad (7.2)$$

where K is a complex constant, and this defines ν uniquely, via the combination of (NPj) and (NPn), except in the case $\beta_1=3a\Lambda/4$. We omit the resulting messy formula. If $\beta_1=3a\Lambda/4$, then K must be real but a second constant of integration still appears in ν. These solutions appear as the Einstein spaces on line $A_\alpha(x, y)$ in table 1 of ref. [15].

The group is of type VI_h, where $h=-[a\Lambda/(a\Lambda-2\beta_1)]^2$, unless $\beta_1=a\Lambda/2$, when it is of type IV, and there are three essential parameters: Λ, a, two in K (or K and the constant in ν if $\beta_1=3a\Lambda/4$) and β_1, less the boost and coordinate scaling freedoms. As mentioned above, the special case $\beta_1=0$ (with a group of Bianchi type III) admits as maximal symmetry group a G_4. The case $\beta_1=5a\Lambda/4$, with K real, gives the type N homogeneous solution written in terms of a group of type $\mathrm{VI}_{-(2/3)^2}$.

When $\beta_1=a\Lambda$, eq. (3.31) gives $\mu=0$ and (3.30) becomes $\gamma=\gamma_0/f$. Ψ_4 is still given by (7.2), but (NPn) and (NPj) imply that K is real. The group is of type III, and there is one essential parameter: Λ, a, K, and γ_0 less the null-rotation, boost and coordinate scaling freedoms.

7.3. PETROV TYPE N SOLUTIONS WITH $\tau+\pi=0=\mathrm{Im}(\pi)$

As $a=0$, β_1 is no longer constrained by (7.1). As in section 7.2, if $\beta_1\neq 0(=a\Lambda)$, $c=0$ and we can use the null-rotation freedom to set $\gamma=\mu=0$. In this case (B8) yields

$$\Psi_4=Kx^{-3}\mathrm{e}^{-4\mathrm{i}\beta_1/\Lambda x}$$

[cf. eq. (7.2)], where K is a complex constant, and ν can be found from (NPj)+(NPn). The group has type VI_0, and there are two essential parameters: Λ, K and β_1 less the boost and scaling freedoms. These solutions appear as the Einstein spaces on line $A(x)\mathrm{e}^{y}$ in table 1 of ref. [15].

The other case, $\beta_1=0$, is a specialization of section 6.1: $\Psi_4=K/x^3$, where K is a real constant, and we must take $t=0$, $\nu_0=K/2p$. These solutions appear as the Einstein spaces on line $u^{-2\beta-2}A(xu^{\beta})$ in table 1 of ref. [15]. The group types possible are just as in section 6.1, and, with K replacing t, so is the parameter count. These solutions all admit a G_4, the additional symmetry being a null rotation in the orbit of the G_3, and are four-dimensionally homogeneous (admitting a G_5) if $\gamma_0=\mu_0=0$, giving the homogeneous solution in the form of a solution with a G_3 of Bianchi type I or II. The cases $5\mu_0=-3\gamma_0$ and $5\mu_0=-2\gamma_0$ also turn out to be homogeneous, presenting the homogeneous solution in the form of one with a group of type VI_h where $h=-16$ or $h=-(7/3)^2$, respectively. (Note that the latter type does not arise in the same way as the group of the same type in the homogeneous type III solution!)

This second subset of solutions are in fact those specializations of the solution of Barnes [9] which admit a G_4 containing a simply transitive G_3. Barnes' solution was given in the form

$$\mathrm{d}s^2=\alpha^2(-2\,\mathrm{d}u\,\mathrm{d}v+\mathrm{d}y^2+2\epsilon\,\mathrm{d}u^2)+\mathrm{d}x^2\ ,$$

where $A=\mathrm{e}^{2cx}/u$, $\epsilon=\epsilon_0(u)\mathrm{e}^{-3xc}$, $c^2=-2\Lambda$ and $\epsilon_0(u)$ is an arbitrary function of u. It is the general solution for an Einstein space submitting a G_3 acting on two-dimensional null surfaces. For certain $\epsilon_0(u)$ it admits a G_4. The relevant metrics are the Einstein space solutions of the forms (C1) and (C2) of ref. [8], which include some metrics with no simply transitive G_3. Using the methods described in ref. [20] we have been able to identify the various subcases in ref. [8], including those where the G_4 does not contain a simply transitive G_3.

7.4. THE HOMOGENEOUS PETROV TYPE N SOLUTION

The four-dimensionally homogeneous metric is, like the type III homogeneous case, due to Kaigorodov [24], and is eq. (10.33) in ref. [6] and eq. (4.8) in ref. [10]. The full group is a G_5. Following the methods of ref. [25], we find the following subgroups G_3: groups of type $\mathrm{VI}_{-1/9}$ acting on spacelike surfaces (as found in ref. [10]), and groups of types I, II, $\mathrm{VI}_{-(2/3)^2}$, $\mathrm{VI}_{-(7/3)^2}$ and VI_{-16} acting on timelike surfaces containing the repeated principal null direction. All these groups, except for those of type $\mathrm{VI}_{-(2/3)^2}$, arise from the specializations described in section 7.3, the remaining cases arising in section 7.2. The solution admits a G_3 acting on two-dimensional null surfaces and is a special case of Barnes' solution (see section 7.3) and of ref. [15]. As with the Petrov type III homogeneous solution, all its manifestations among the solutions found here occur as special cases of a multi-parameter family.

7.5. PETROV TYPE N (VACUUM) SOLUTIONS WITH $\tau=\pi=0$

These vacuum solutions are all in principle known (due to Kundt), and are pp-waves (see ref. [6], section 21.5). All we are doing is thus specifying which of the known solutions have a G_3 acting on timelike surfaces containing the repeated PND. The coordinates of the previous sections no longer work; instead we choose an x coordinate so that $\delta+\bar{\delta}=2\partial_x$. The equations of the first half of section 3 still hold, but the integrations must be redone. To start with, (NPl) leads to two possibilities: either $\beta=0$ or $\mathrm{Re}(\beta)=1/2x$.

The first of these possibilities leads to plane waves (see section 27.5 of ref. [6]), and a short calculation shows these have groups G_3 on timelike surfaces of Bianchi type VI_h, VI_0, IV, V, II or I; Harness [14] recovered the first four of these forms.

The second possibility corresponds to the second case discussed in ref. [6], section 27.5.1. We have $\beta=1/2x+\mathrm{i}\beta_1/x$, and integrating (NPm) gives $\mu=(\mu_0+\mathrm{i}\gamma_1)/x$. Then (3.9) determines γ up to a real constant: $\gamma=\mu+\gamma_0$ where $\mu_0+2\beta_1\gamma_1=0$, according to (3.7). We can always use a null rotation to set $\gamma_1=0$, in which case $\mu_0=0$ (and hence $\mu=0$) also. The only remaining constraint $\gamma_0\beta_1=0$ comes from the Jacobi identity (3.8). Finally, we can integrate (NPn) and (NPj):

$$\Psi_4=Kx^{-2-4\mathrm{i}\beta_1}\,,\qquad \nu=-\Psi_4/(2+8\mathrm{i}\beta_1)\,,$$

for some complex constant K. The group is of type VI_0 if $\gamma_0\neq 0$ or II otherwise. There are two essential parameters: γ_0 or β_1 and K less the boost.

8. The metric

All our solutions must be in Kundt's class (see ref. [6], ch. 27) but could in most cases only be given implicitly or in a messy way in the canonical coordinates of ref. [6], eq. (27.7). The coordinate choice (3.11) corresponds instead to a variable like the z of ref. [6], eq. (27.41), used to express the type D metrics in

Kundt's form *. The type D and type N vacuum metrics are already covered by the metric forms of ref. [6], section 27, so we have only to give a metric form for the cases with $\epsilon=0\neq\pi$.

It is guaranteed, since we have solved the NP equations, that we can find a metric form (see section 2.2); starting from (3.11), and introducing coordinates successively for D, $\delta-\bar{\delta}$ and Δ, we find we can cover all cases by

$$\begin{aligned}
\boldsymbol{k} &= \sqrt{x^2+a^2}\ \mathrm{e}^{4ky}\,\mathrm{d}u\,,\\
\boldsymbol{l} &= \sqrt{x^2+a^2}\ [\mathrm{d}v+(4\beta_1 v-G)\,\mathrm{d}y\\
&\quad +(H+G^2/4\Lambda+2\mu_0 Gy/p+2aqvG+2p\nu_1 y+2\gamma_0 v/p-4akqv^2)\,\mathrm{e}^{4ky}\,\mathrm{d}u]\,,\\
\boldsymbol{m} &= \mathrm{d}x/2f+\mathrm{i}f[\mathrm{d}y+(-G/2\Lambda-2aqv-2\mu_0 y/p)\,\mathrm{e}^{4ky}\,\mathrm{d}u]\,,
\end{aligned} \tag{8.1}$$

where $k=\beta_1-(1-q)a\Lambda$, $q=0$ if $\tau+\bar{\pi}=0$ and $q=1$ if $\tau+\pi=0$, and G and H are functions of x related to the spin coefficients by

$$G=\mathrm{i}\int\frac{(\mu-\bar{\mu})\,\mathrm{d}x}{\sqrt{x^2+a^2}}\,,\qquad H=\int\frac{(\nu+\bar{\nu})\,\mathrm{d}x}{2f}\equiv\int N/f^2\,\mathrm{d}x\,.$$

This possibility is connected with theorem 27.1 of ref. [6], which shows how different solutions in Kundt's class are related by functions obeying certain linear equations. G is always taken to be 0 if $\mu=0$, and in fact is chosen non-zero exactly in the Petrov type III metrics.

The terms in (8.1) in which γ_0, μ_0 and ν_1 appear explicitly can be eliminated, respectively, by choice of the constant in G and a coordinate transformation if $\beta_1+qa\Lambda\neq0$, by the same method if $\beta_1\neq0$ also, and by choice of the constant in H if $\beta_1+k\neq0$. The exceptional cases requiring all three parameters are thus those of section 7.1 with $a=0$ and of section 7.3 with $\beta_1=0$. γ_0 and ν_1 are also required in the cases of section 4 with $\beta_1=0$, and γ_0 in the cases of section 7.2 with $\beta_1=a\Lambda$ (taking $G=0$).

Finally, the function H has the value $-(n+2aG^2)/4\beta_1$ [see (3.17)] in the following cases: section 6.1 if $a\neq0$; section 7.1; section 7.3 if $\beta_1\neq0$; and section 4 if $\beta_1\neq0$ [H then obeys (4.2)]. In section 6.2, $H=-(n+3aG^2/2)/6\Lambda a$. In section 7.2, $n=H(2a\Lambda-4\beta_1)$ (which has a special case when $\beta_1=a\Lambda/2$). These results arise from the coupling of equations for N and n in (NPn) [see, for example, (4.1)], and only in the exceptional cases where this coupling is broken does one have to integrate for H separately from n. These exceptions are: section 4 with $\beta_1=0$; section 7.1 with $a=0$; section 7.2 with $\beta_1=a\Lambda/2$; and section 7.3 with $\beta_1=0$.

9. Summary and concluding remarks

Most of this paper has been an exercise in deriving solutions using the Newman–Penrose formalism in the now classic manner. As a result of the calcu-

* This relation is misprinted in ref. [6]: it should read $P^2\mathrm{d}z=\mathrm{d}x$.

lations here together with those of ref. [10] and earlier results given in ref. [6] we can now present a complete list of all solutions covered by our title. We have

1. the spaces of constant curvature: Minkowski, de Sitter and anti de Sitter, each conformally flat and with a G_{10}, containing numerous G_3;
2. homogeneous plane waves, with $\Lambda=0$, of Petrov type N with a G_6 containing G_3 of types VI, IV or VII;
3. the metric of Bertotti–Robinson form, of Petrov type D with a G_6 containing G_3 of type III;
4. the homogeneous Petrov type N solution with $\Lambda<0$ and a G_5 containing various G_3 (see section 7.4);
5. the homogeneous Petrov type III solution with $\Lambda<0$ and a G_4 containing various G_3 (see section 6.3);
6. inhomogeneous plane waves of Petrov type N and within a G_5;
7. the generalized Taub–NUT solutions with a G_4 acting on spacelike surfaces and various G_3 subgroups [6,10];
8. a Petrov type III solution with a G_3 of type $\mathrm{VI}_{-1/9}$ [10];
9. Leroy's twisting Petrov type N solution with a G_3 of type $\mathrm{VI}_{-1/9}$ [10];
10. the twisting Petrov type III solution with a G_3 of type $\mathrm{VI}_{-1/9}$ [10];
11. a Petrov type II solution in Kundt's class with a G_3 of type $\mathrm{VI}_{-1/9}$ [10];
12. Petrov type D solutions analogous to the Taub–NUT solutions (see section 5);
13. for each Λ, a three-parameter family of Petrov type II solutions with a G_3 of type VIII (see section 4);
14. for each Λ, a two-parameter family of Petrov type II solutions with a G_3 of type VI_0 (see section 4);
15. for each Λ, a three-parameter family of Petrov type II solutions with G_3 generally of type VI_h or VII_h (see section 4);
16. for each Λ, a two-parameter family of Petrov type II solutions with a G_3 in general of type III (see section 4);
17. for each $\Lambda<0$, a two-parameter family of solutions of Petrov type III with a G_3 of type VIII (see section 6.1);
18. for each $\Lambda<0$, a two-parameter family of Petrov type III solutions with a G_3 of one of various types, in general type VI_h (see section 6.1); the family contains the homogeneous solution;
19. for each $\Lambda<0$, a two-parameter family of Petrov type III solutions with a G_3 of type $\mathrm{VI}_{-1/9}$ (see section 6.2); the family contains the homogeneous solution;
20. for each $\Lambda<0$, a solution of Petrov type N with a G_4 containing a G_3 of type VIII and G_3 of type III (see section 7.1);
21. for each $\Lambda<0$, a solution of Petrov type N with a G_3 of type III (see section 7.2);
22. for each $\Lambda<0$, a two-parameter family of Petrov type N solutions with a G_3 of type VI_h (see section 7.2); this family contains the homogeneous solution;

23. for each $\Lambda<0$, a one-parameter family of Petrov type N solutions with a G_3 of type VI_0 (see section 7.3);

24. for each $\Lambda<0$, a two-parameter family of Petrov type N solutions admitting a G_4 containing a simply transitive G_3 of various possible types, or no simply transitive G_3 (see section 7.3); this family contains the homogeneous solution in several forms;

25. a two-parameter family of Petrov type N vacuum pp-wave solutions with a G_3 of type VI_0 (see section 7.5);

26. a two-parameter family of Petrov type N vacuum pp-wave solutions with a G_3 in general of type VI_0 (see section 7.5).

It is worth remarking that the set of all Einstein spaces with hypersurface homogeneity has only five essential parameters (including Λ), so the existence of two four-parameter type II families shows that the restriction to algebraically special solutions loses only one degree of freedom.

It is apparent that there are strong similarities between the metrics of differing Petrov types found here [since they are all included in (8.1)]. For example, we think it highly probable that one can pass from the type II solutions to the similar type III and type N solutions by a limiting procedure, with an infinitely large null rotation compensating for the m and l of (3.13) tending to 0. The solutions are also related in the way described by ref. [6], theorem 27.1. We have not worked out the details of these relationships.

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Appendix. The NP equations

$$\delta\epsilon=\epsilon(\beta-\bar{\beta}-\bar{\pi}) , \tag{NPe}$$

$$0=2(\bar{\tau}+\pi)\beta-3\epsilon\gamma-\bar{\gamma}\epsilon+\tau\pi+\Psi_2-\Lambda , \tag{NPf}$$

$$-\delta\pi=(\pi-\beta-\bar{\beta})\pi+2\epsilon\mu , \tag{NPg}$$

$$-\delta\pi=\pi(\bar{\pi}+\beta+\bar{\beta})-2\epsilon\mu+\Psi_2+2\Lambda , \tag{NPh}$$

$$0=2(\pi+\bar{\tau})\mu+(\gamma-\bar{\gamma})\pi-4\epsilon\nu+\Psi_3 \tag{NPi}$$

$$-\delta\nu=(\mu+\bar{\mu})\mu+(3\gamma-\bar{\gamma})\mu+(-3\beta+\bar{\beta}+\pi-\bar{\tau})\nu-\Psi_4 , \tag{NPj}$$

$$-2\delta\beta=2\beta\bar{\beta}+2\beta^2+\epsilon(\gamma-\bar{\gamma})-\Psi_2+\Lambda , \tag{NPl}$$

$$-2\delta\mu=\pi(\gamma-\bar{\gamma})+2\mu(\beta+\bar{\beta})-\Psi_3\ , \tag{NPm}$$

$$\delta\nu=\mu^2+\mu\bar{\mu}+(\gamma+\bar{\gamma})\mu-\bar{\nu}\pi+(\tau-3\beta+\bar{\beta})\nu\ , \tag{NPn}$$

$$\delta\gamma=(\tau+\bar{\beta}-\beta)\gamma+\mu\tau-\epsilon\bar{\nu}+2\beta\bar{\mu}\ , \tag{NPo}$$

$$\delta\tau=(\tau+\beta+\bar{\beta})\tau\ , \tag{NPp}$$

$$-\delta\tau=(\bar{\beta}+\beta-\bar{\tau})\tau-\Psi_2-2\Lambda\ , \tag{NPq}$$

$$-\delta\gamma=\epsilon\nu+(\tau+\beta)\mu-(\bar{\gamma}-\bar{\mu})\beta+(\bar{\beta}-\bar{\tau})\gamma-\Psi_3\ , \tag{NPr}$$

$$3\rho\Psi_2=0\ , \tag{B3}$$

$$\delta\Psi_2=3\tau\Psi_2\ , \tag{B4}$$

$$-\delta\Psi_2=-2\epsilon\Psi_3+3\pi\Psi_2\ , \tag{B5}$$

$$-\delta\Psi_3=2(\beta-\tau)\Psi_3-3\mu\Psi_2\ , \tag{B6}$$

$$\delta\Psi_3=4\epsilon\Psi_4-2(2\pi-\beta)\Psi_3-3\mu\Psi_2\ , \tag{B7}$$

$$-\delta\Psi_4=(4\beta-\tau)\Psi_4-2(2\mu+\gamma)\Psi_3+3\nu\Psi_2\ , \tag{B8}$$

$$[\Delta,\delta-\bar{\delta}]=(\bar{\nu}-\nu)D-(\tau-\bar{\tau}-2(\beta-\bar{\beta}))\Delta-(\mu+\bar{\mu})(\delta-\bar{\delta})\ ,$$

$$[\delta-\bar{\delta},D]=(\pi-\bar{\pi}+2(\beta-\bar{\beta}))D\ ,$$

$$[D,\Delta]=-(\gamma+\bar{\gamma})D-(\epsilon+\bar{\epsilon})\Delta+(\pi+\bar{\tau})(\delta-\bar{\delta})\ ,$$

$$[\delta+\bar{\delta},\Delta]=-(\nu+\bar{\nu})D+(\tau+\bar{\tau})\Delta-(\gamma-\bar{\gamma})(\delta-\bar{\delta})\ ,$$

$$[\delta+\bar{\delta},D]=-(\pi+\bar{\pi})D\ ,$$

$$[\delta+\bar{\delta},\delta-\bar{\delta}]=2(\gamma-\bar{\gamma})D-2(\beta+\bar{\beta})(\delta-\bar{\delta})\ .$$

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